and Newton's second law for one turn of the slinky ($dn=1$) yields

$$\kappa \frac{\partial^2 x}{\partial n^2}+mg=m \frac{\partial^2 x}{\partial t^2}. \tag{A3}$$

The static solution to Eq. (A3) satisfying $x(0,0)=0$ is given by

$$x_s=-\frac{mg}{2\kappa} n^2+bn, \tag{A4}$$

where b is an arbitrary constant. The physically meaningful values for n in the above formula are those for which the tension is positive. Using Eq. (A1) the tension is given by

$$T=-mgn+\kappa b. \tag{A5}$$

The boundary condition $T=0$ for $n=N$ determines b for the free slinky while $T=F$ yields b for the stretched slinky. All of the formulas for the static situation may be obtained once b is defined.

To obtain the equation for longitudinal waves we may expand $x(n,t)$ about the static equilibrium position and write

$$x(n,t)=u(n,t)+x_s(n). \tag{A6}$$

Substituting Eq. (A6) into Eq. (A3) then results in the wave equation (2.14) from which the normal mode development follows.

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The charge densities in a current-carrying wire

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In the lab frame the total linear charge density of a current-carrying wire must be zero, while in the rest frame of the electrons making up the current the total volume charge density must be zero. These two pieces of information enable the determination of the volume, surface, and linear charge densities of such a wire in both of these frames using only straightforward relativistic length contractions and simple mathematics.

I. INTRODUCTION

All magnetic fields and forces acting on an electrical charge may be understood in terms of electric fields and forces felt by the charge in its own rest frame. This view, developed by Feynman, Leighton, and Sands,[1] Purcell,[2] and others[3] essentially describes magnetic phenomena as relativistic effects. The case most often considered is the Lorentz force felt by a charge moving through the magnetic field of a wire carrying constant current. The usual approach is to model the current-carrying wire as a superposition of two uniform charge densities: one positive (the rigid ion lattice) and one negative (the flowing conduction electrons). By transforming these two charge densities into the rest frame of a test charge q moving relative to the wire, it is possible to show that the magnetic force felt by q in the lab frame is equivalent to an electrostatic force in q's rest frame. Although Purcell's and Feynman's approaches to this question are quite similar, Purcell frames the problem in terms of *linear* charge densities while Feynman frames it in terms of *volume* charge densities.

It is a well-known fact that a current-carrying wire is neutral in the rest frame of the lattice of ions making up the wire, i.e., in the lab frame. More specifically, it is the total *linear* charge density of the wire which is zero in this frame. This may be understood as a consequence of the availability of free electrons in the electron source and sink associated with the current. Free electrons in the source and sink are at rest in the lab frame, and if the wire is not neutral in this frame these electrons will move to make it so. Matzek and Russell[4] and later Peters[5] showed that if a three-dimensional wire is considered, it is found that the distribution of charge with radius in a cylindrical wire is

not uniform, and that the wire has separate volume and surface charge densities. This indicates that Feynman's direct generalization of the linear charge densities of a wire to volume charge densities was not appropriate.

The nonuniformity in the radial charge distribution in a current-carrying wire is such that there is a concentration of negative charge toward the center of the wire. The apparent mechanisms causing this nonuniformity are different in the lab frame S and the rest frame of the electrons making up the current, S'. In S, the mechanism has been described as a "pinch effect"[4] or a "self-induced Hall effect":[5] the electrons making up the current are moving through a magnetic field caused by their fellow electrons, and so feel a force of magnitude $ev_dB(r)$ toward the center of the wire. Here e is the magnitude of the electron charge, v_d is the drift velocity associated with the current, and $B(r)$ is the magnetic field strength at a radius r in the wire. In equilibrium, the electrons will concentrate toward the center of the wire until the electric force caused by the resulting radial charge separation is balanced by the Lorentz force $ev_dB(r)$. Thus, even though the *linear* charge density in S, the lab frame, is zero, it is due to the superposition of a negative volume charge density and a positive surface charge density.

In S', the electrons making up the current are at rest, and so do not feel any magnetic force; however, the lattice of positive ions appears to be Lorentz contracted, and so there is initially an excess of positive charge in the bulk of the wire. These electrons at rest in S' therefore experience an electrostatic force toward the center of the wire and concentrate toward the center until this force is balanced by the opposing force due to the resulting radial charge separation. One way to view this situation is to consider a cylindrical Gaussian surface with some radius less than the radius of the wire; because of the length contraction of the positive lattice, this Gaussian surface will initially contain a net positive charge, which results in an electric field pointing radially outward from the center of the wire. When equilibrium is reached, the volume charge density of the wire in S' is zero and a thin layer of positive surface charge density is left behind by the migration of electrons toward the wire center. This mechanism for the generation of the surface charge density in the S' frame was apparently not recognized by previous workers on this topic.[6]

We see that in both S and S', a positive surface charge density is induced, but in S the total *linear* charge density is zero, while in S' the total *volume* charge density is zero. This apparent asymmetry is due to the fact that in S free electrons in the electron source and sink which are driving the current will move so as to maintain neutrality of the wire as a whole, while in S' the free electrons in the wire will move so as to maintain neutrality of the *bulk* of the wire.

Peters derived correct expressions for the volume charge densities in both S and S', and for the surface charge density in S; although he also questioned the existence of this surface charge density, due to lack of a mechanism for its generation in S'. Hernandez *et al.*[7] later showed that the deflection of electrons associated with the "self-induced Hall effect" in S must occur in the S' frame as well, but agreed with Peters that this may not necessarily lead to a surface charge density, and did not derive an expression for it. In the following section we calculate the volume, surface, and linear charge densities in both the S and S' frames, using only the transformation properties of the linear and volume charge densities, and the information above about the linear charge density in S and the volume charge density in S'. In addition to the derivations here being quite simple, they are the first such derivations done completely within the context of magnetic phenomena as relativistic effects.[8]

II. THE LINEAR, VOLUME, SURFACE CHARGE DENSITIES IN S AND S'

We denote linear charge densities by λ, volume charge densities by ρ, and surface charge densities by σ. Quantities in S will be "unprimed," while those in S' will be marked with a "prime" ($'$). We consider a cylindrical wire of radius a which is composed of a rigid lattice of ions and a nonrigid distribution of electrons which make up the current. In frame S, the lab frame, the ion lattice is at rest and the negative charge distribution appears Lorentz contracted by a factor $\gamma=(1-v_d^2/c^2)^{1/2}$, where v_d is the drift velocity associated with the current and c is the speed of light. In frame S', the rest frame of the moving electrons, the ion lattice appears contracted by the same factor. The radius a is the same in the two frames, since the drift velocity has no component in the radial direction. From the arguments in Sec. I, we require that the total linear charge density λ in S be zero, and that the total volume charge density ρ' in S' be zero.

Let ρ_0 be the volume charge density of the ion lattice in S. The volume charge density of the lattice in S', ρ'_+, will simply be $\gamma\rho_0$; since the length of the lattice is contracted, the charge density is enhanced. Therefore, in order for the net volume charge density in S' to be zero the negative volume charge density ρ'_-, must be

$$\rho'_-=-\rho'_+=-\gamma\rho_0. \tag{1}$$

We may find ρ_- immediately, since ρ_- will simply be the negative volume charge density in S' enhanced by the Lorentz contraction of the electron distribution as seen in S

$$\rho_-=\gamma\rho'_-=-\gamma^2\rho_0\,.$$

The net volume charge density in the lab frame S is[9]

$$\rho\equiv\rho_++\rho_-=\rho_0(1-\gamma^2),\quad \therefore \rho=-\rho_0\beta_d^2\gamma^2, \tag{2}$$

where $\beta_d=v_d/c$; this is negative, as expected.

We may now use the requirement that λ be zero to determine the surface charge density σ in S. We may express λ in terms of σ and ρ by considering a small length of wire dl, which has volume $\pi a^2 dl$ and surface area $2\pi a\, dl$. This length of wire contains a small amount of charge dq, which is zero since λ is zero. The surface charge density σ is found to be[10]

$$dq=\lambda dl=\rho\cdot(\pi a^2 dl)+\sigma\cdot(2\pi a\, dl)=0,$$

$$\lambda=\rho\cdot\pi a^2+\sigma\cdot 2\pi a=0,$$

$$\sigma=-\rho a/2,\quad \therefore \sigma=\rho_0 a\beta_d^2\gamma^2/2, \tag{3}$$

where the assumption has been made that the region of excess positive charge density near the surface of the wire is very thin, which is most definitely the case for typical current drift velocities, as pointed out by Matzek and Russell.[4]

Let us now find the surface charge density in S'. Consider first the positive linear charge density λ'_+, which will have contributions from both volume and surface charge densities (since there is no negative charge density, $\sigma_+ = \sigma$)

$$\lambda'_+ = \rho'_+ \cdot \pi a^2 + \sigma' \cdot 2\pi a = \gamma \lambda_+ ,$$

where $\lambda_+ = \rho_+ \cdot \pi a^2 + \sigma \cdot 2\pi a$. Substituting for ρ_+ and ρ'_+ in terms of ρ_0,

$$\gamma \rho_0 \cdot \pi a^2 + \sigma' \cdot 2\pi a = \gamma [\rho_0 \cdot \pi a^2 + \sigma \cdot 2\pi a], \quad \therefore \sigma' = \gamma \sigma. \tag{4}$$

In other words, the ion and electron *volume* charge densities in one frame must be transformed separately and superposed to obtain the net volume charge density in the other frame, but the *surface* charge density, since it is composed of charge of only one sign, transforms directly, as expected. Lastly, the total linear charge seen in S' may now be found (remember that in this frame, there is no net volume charge density):

$$\lambda' = \sigma' \cdot 2\pi a, \quad \therefore \lambda' = \pi a^2 \rho_0 \beta_d^2 \gamma^3. \tag{5}$$

III. CONCLUSION

We have determined the surface, volume, and linear charge densities of a cylindrical current-carrying wire in both the lab frame S and in the rest frame of the electrons constituting the current, S', and have identified a mechanism for the generation of the surface charge density in S'. The derivations were based on the knowledge that in the lab frame the total *linear* charge density is zero, while in the rest frame of the moving conduction electrons the total *volume* charge density is zero. Both of these conditions may be understood as consequences of the simple fact that in any frame in which there are free electrons at rest, these electrons will arrange themselves in such a way as to minimize any net charge density in that frame. The derivations were done using only very simple mathematics and basic length contractions. This approach may help students to develop a deeper understanding of the steady-state charge distribution in a current-carrying wire as viewed from difference reference frames, and more generally, an appreciation of magnetic fields and forces as relativistic phenomena.

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[9]Equation (2) corresponds to Eq. (3) in Peters' article.

[10]Equation (3) corresponds to Eq. (5) in Peters' article.

PARTICLE PHYSICISTS' GOSSIP

A physicist joked to me that this talk among physicists was like "photons being exchanged among interacting particles." Exchanging judgments about one's peers, persuading colleagues to support one's work, managing news, being a competent performer of combative, tendentious jokes (preferably using technical language from particle physics to describe human behavior), and being an informed gossip are crucial skills for a successful particle physicist.

Sharon Traweek, *Beamtimes and Lifetimes: The World of High Energy Physicists* (Harvard U. P., Cambridge, 1988), p. 121.